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Stability estimate for a multidimensional inverse spectral problem with partial spectral data

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ABSTRACT

In Bellassoued, Choulli and Yamamoto (2009) [4] we proved a log–log type stability estimate for a multidimensional inverse spectral problem with partial spectral data for a Schrödinger operator, provided that the potential is known in a small neighbourhood of the boundary of the domain. In the present paper we discuss the same inverse problem. We show a log type stability estimate under an additional condition on potentials in terms of their X-ray transform. In proving our result, we follow the same method as in Alessandrini and Sylvester (1990) [1] and Bellassoued, Choulli and Yamamoto (2009) [4]. That is we relate the stability estimate for our inverse spectral problem to a stability estimate for an inverse problem consisting in the determination of the potential in a wave equation from a local Dirichlet to Neumann map (DN map in short).

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be a bounded domain with smooth boundary Γ and let γ be a closed subset of Γ with nonempty interior.

If $q \in L^\infty(\Omega)$, we denote by A_q the unbounded operator $A_q = -\Delta + q$, having the domain $\mathscr{D}(A_q) = H_0^1(\Omega) \cap H^2(\Omega)$. It is well known that the spectrum of A_q consists in a sequence of eigenvalues, counted according to their multiplicity,

$$-\infty < \lambda_{1,q} \leqslant \lambda_{2,q} \leqslant \cdots \leqslant \lambda_{k,q} \to +\infty.$$

The corresponding eigenfunctions are denoted by $(\varphi_{k,q})$. We make the assumption that the sequence $(\varphi_{k,q})$ form an orthonormal basis of $L^2(\Omega)$. That is

$$\|\varphi_{k,q}\|_{L^2(\Omega)} = 1.$$

Unless otherwise stated, C denotes a generic positive constant depending only on Ω, γ and q.

Since $\varphi_{k,q}$ is the solution of the following boundary value problem (BVP in short)

$$\begin{cases} (-\Delta + q)\varphi = \lambda_{k,q}\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \Gamma, \end{cases}$$

we derive from the classical H^2 a priori estimates

$$\|\varphi_{k,q}\|_{H^2(\Omega)} \leqslant C\lambda_{k,q}\|\varphi_{k,q}\|_{L^2(\Omega)} = C\lambda_{k,q}.$$

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Therefore

$$\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\Gamma)} \leqslant C\lambda_{k,q}.$$

But there exists a constant $K \geqslant 1$ such that

$$K^{-1}k^{\frac{2}{d}} \leqslant \lambda_{k,q} \leqslant Kk^{\frac{2}{d}}. \tag{1.1}$$

Consequently,

$$\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\gamma)} \leqslant C\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\Gamma)} \leqslant Ck^{\frac{2}{d}}. \tag{1.2}$$

We fix $d/2+1<\zeta \leqslant d+1$. It follows from (1.2) that the sequence $(k^{-\frac{2\zeta}{d}}\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\gamma)}) \in \ell^1$. We recall that ℓ^1 is the usual Banach space of real-valued sequence such that the corresponding series is absolutely convergent. This space is equipped with its natural norm.

Let $w=(w_k)$ be the sequence given by $w_k = k^{-\frac{2\zeta}{d}}$, for each $k \geqslant 1$. We then introduce the following Banach space

$$\ell^1\big(H^{\frac{1}{2}}(\gamma), w\big) = \big\{g=(g_k);\ g_k \in H^{\frac{1}{2}}(\gamma),\ k \geqslant 1, \text{ and } \big(w_k\|g_k\|_{H^{\frac{1}{2}}(\gamma)}\big) \in \ell^1\big\}.$$

The natural norm on this space is

$$\|g\|_{\ell^1(H^{\frac{1}{2}}(\gamma),w)} = \sum_{k\geqslant 1} w_k\|g_k\|_{H^{\frac{1}{2}}(\gamma)}.$$

Let $\mu=(\mu_k)$ be the sequence of eigenvalues of A_0 (that is A_q with $q=0$). As a consequence of the min–max formula (e.g. [10]), we have

$$|\lambda_{k,q}-\mu_k| \leqslant \|q\|_{L^\infty(\Omega)}, \quad k \geqslant 1.$$

In the other words $\lambda_q = (\lambda_{k,q})$ belongs to the affine space $\tilde{\ell}^\infty = \mu + \ell^\infty$, where ℓ^∞ is the usual Banach space of bounded real-valued sequences. We equip $\tilde{\ell}^\infty$ with the following distance

$$d_\infty(\lambda_1,\lambda_2) = \big\|(\lambda_1-\mu)-(\lambda_2-\mu)\big\|_{\ell^\infty} = \|\lambda_1-\lambda_2\|_{\ell^\infty},$$

if $\lambda_i \in \tilde{\ell}^\infty$, $i=1,2$.

We fix a parameter $\epsilon>0$ and we set

$$\Omega_\epsilon = \big\{x \in \mathbb{R}^d \setminus \overline{\Omega};\ \operatorname{dist}(x,\Omega)<\epsilon\big\}.$$

We denote the line segment in the direction $\theta \in \mathbb{S}^{d-1}$ and passing through $x \in \mathbb{R}^d$ by $\ell(x,\theta)$, i.e.

$$\ell(x,\theta) = \big\{y \in \mathbb{R}^d;\ y = x+t\theta,\ t \in \mathbb{R}\big\}.$$

Let Γ_1 be an open subset (possibly empty) of Γ, $\Omega_{0,\epsilon}$ be an open subset of Ω_ϵ and $S \subset \mathbb{S}^{d-1}$. We assume that Γ_1, $\Omega_{0,\epsilon}$ and S can be chosen in such way that

$$\ell(x,\theta)\cap\Gamma_1 = \emptyset, \quad \text{for all } x \in \Omega_{0,\epsilon},\ \theta \in S. \tag{1.3}$$

Next, we fix $q_0 \in \mathcal{C}^{0,\mu}(\overline{\Omega})$, $0<\mu<1$, and we consider the set

$$\mathfrak{X}(M,\omega) = \big\{q \in \mathcal{C}^{0,\mu}(\overline{\Omega});\ \|q\|_{L^\infty(\Omega)} \leqslant M,\ q(x)=q_0(x) \text{ in } \omega\big\}, \tag{1.4}$$

where $\mathcal{C}^{0,\mu}(\overline{\Omega})$ is the usual Hölder space, $\omega \subset \Omega$ is an arbitrary neighbourhood of Γ in Ω and M is a given positive constant.

We recall that the X-ray transform of a function f, defined on $\mathbb{R}^d$, is formally given by

$$\mathcal{P}f(\theta,x) = \int_{\mathbb{R}} f(x+t\theta)\,dt, \quad \theta \in \mathbb{S}^{d-1},\ x \in \mathbb{R}^d.$$

Let $\mathfrak{X}(M,\omega,S,\Omega_{0,\epsilon})$ denote the set of all $(q_1,q_2) \in \mathfrak{X}(M,\omega)\times\mathfrak{X}(M,\omega)$ satisfying the following estimate

$$\|\mathcal{P}q\|_{L^\infty(\mathbb{S}^{d-1}\times\Omega_\epsilon)} \leqslant K\|\mathcal{P}q\|_{L^\infty(S\times\Omega_{0,\epsilon})}, \tag{1.5}$$

for some positive constant K, where $q=q_1-q_2$ is extended by zero outside Ω.

To our knowledge, the first stability estimate for a multidimensional inverse spectral problem is due to Alessandrini and Sylvester [1]. In [1] the authors studied the case when the full spectral data is given. The case of partial spectral data was considered by the authors in [4] where we establish a log–log type stability estimate provided that the potentials are known near the boundary. The present work is the continuation of [4]. We show a log type stability estimate under an additional condition on potentials in terms of their X-ray transform which is characterized by $\mathfrak{X}(M,\omega,S,\Omega_{0,\epsilon})$. Precisely, we prove the following theorem.

Theorem 1.1. *We assume that $\gamma = \Gamma \setminus \Gamma_1$. Let $0 \leqslant \alpha < \frac{1}{2}$. Then there exist $C > 0$ and $\delta \in (0,1)$ such that the following estimate holds*

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leqslant C\big(\big(\eta + \eta^\theta\big)^\delta + \big|\log C\big(\eta + \eta^\theta\big)\big|^{-\delta}\big) \tag{1.6}$$

for any $(q_1, q_2) \in \mathfrak{X}(M, \omega, S, \Omega_{0,\epsilon})$, where

$$\eta = d_\infty(\lambda_{q_1}, \lambda_{q_2}) + \|\partial_\nu \varphi_{q_1} - \partial_\nu \varphi_{q_2}\|_{\ell^1(H^{\frac{1}{2}}(\gamma), w)},$$

$\partial_\nu \varphi_{q_i} = (\partial_\nu \varphi_{k,q_i})$, $i = 1, 2$, and $\theta = 1 - \frac{4(d+1)}{(1-2\alpha)+4(d+1)}$.

If in addition $q_1, q_2 \in H^s(\Omega)$, for $s > d/2$ and $\|q_j\|_{H^s(\Omega)} \leqslant M$, $j = 1, 2$, then there exists $\tau \in (0,1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\big(\big(\eta + \eta^\theta\big)^\tau + \big|\log C\big(\eta + \eta^\theta\big)\big|^{-\tau}\big). \tag{1.7}$$

Assumption (1.5) is technical but essential in our approach for proving a log type stability estimate for the above mentioned inverse spectral problem. We will discuss in the next section two examples of functions (q_1, q_2) belonging to the class $\mathfrak{X}(M, \omega, S, \Omega_{0,\epsilon})$.

2. Examples

We start with an example in the two-dimensional case.

Example 2.1. We identify $\mathbb{R}^2$ and $\mathbb{C}$. Let Ω be the unit ball and

$$\Omega_{0,\epsilon} = \big\{z \in \mathbb{C};\ 1 < |z| < 1 + \epsilon,\ -\pi/4 < \arg(z) < \pi/4\big\},$$
$$S = \big\{z \in \mathbb{C};\ |z| = 1,\ \pi/4 < \arg(z) < 3\pi/4\big\},$$
$$\Gamma_1 = \big\{z \in \mathbb{C};\ |z| = 1,\ 3\pi/4 < \arg(z) < 5\pi/4\big\}.$$

In this case an elementary calculation shows that (1.3) is satisfied.

Proposition 2.1. *Let S and $\Omega_{0,\epsilon}$ be as in Example 2.1. If $(q_1, q_2) \in \mathfrak{X}(M, \omega) \times \mathfrak{X}(M, \omega)$ is such that $q_1 - q_2$, extended by zero outside Ω, is radially symmetric, then $(q_1, q_2) \in \mathfrak{X}(M, \omega, S, \Omega_{0,\epsilon})$.*

Proof. Let $(q_1, q_2) \in \mathfrak{X}(M, \omega) \times \mathfrak{X}(M, \omega)$ be given, $q = q_1 - q_2$, extended by zero outside Ω. We assume that q is radially symmetric and we set

$$A = \big\|\mathcal{P}(q_1 - q_2)\big\|_{L^\infty(S \times \Omega_{0,\epsilon})}.$$

In the sequel we shall identify $q(x, y)$, $(x, y) \in \mathbb{R}^2$ with $q(z)$, $z \in \mathbb{C}$.

Let $k \geqslant 2$ be an integer and $z_0 = re^{i\varphi} \in \Omega_{0,\epsilon}$, $1 < r < 1 + \epsilon$ and $-\frac{\pi}{4} < \varphi < -\frac{\pi}{4} + \frac{\pi}{2k}$. Since q is radially symmetric, we have

$$\left|\int_{\mathbb{R}} q\big(z_0 + te^{i\psi}\big)\,dt\right| = \left|\int_{\mathbb{R}} q\big(e^{\frac{\pi}{2k}}\big(z_0 + te^{i\psi}\big)\big)\,dt\right| \leqslant A, \quad \frac{\pi}{4} < \psi < \frac{3\pi}{4}.$$

That is

$$\left|\int_{\mathbb{R}} q\big(z_1 + te^{i\psi}\big)\,dt\right| \leqslant A, \quad \frac{\pi}{4} + \frac{\pi}{2k} < \psi < \frac{3\pi}{4} + \frac{\pi}{2k},$$

where $z_1 = z_0 e^{\frac{\pi}{2k}} \in \Omega_{0,\epsilon}$. Moreover, from our assumption we have also

$$\left|\int_{\mathbb{R}} q\big(z_1 + te^{i\psi}\big)\,dt\right| \leqslant A, \quad \frac{\pi}{4} < \psi < \frac{3\pi}{4}.$$

Therefore

$$\left|\int_{\mathbb{R}} q\big(z_1 + te^{i\psi}\big)\,dt\right| \leqslant A, \quad \frac{\pi}{4} < \psi < \frac{3\pi}{4} + \frac{\pi}{2k}.$$

We repeat this argument with z_1 in place of z_0. We get, with $z_2 = z_1 e^{\frac{\pi}{2k}} = z_0 e^{2\frac{\pi}{2k}} \in \Omega_{0,\epsilon}$,

$$\left|\int_{\mathbb{R}} q\big(z_2 + te^{i\psi}\big)\,dt\right| \leqslant A, \quad \frac{\pi}{4} < \psi < \frac{3\pi}{4} + 2\frac{\pi}{2k}.$$

We pursue until a final step for which we have

$$\left|\int_{\mathbb{R}} q\big(z_{k-1}+te^{i\psi}\big)\,dt\right| \leqslant A, \quad \frac{\pi}{4}<\psi<\frac{3\pi}{4}+(k-1)\frac{\pi}{2k}=\frac{5\pi}{4}-\frac{\pi}{2k},$$

where $z_{k-1}=z_0e^{(k-1)\frac{\pi}{2k}} \in \Omega_{0,\epsilon}$.

Using again the fact that q is radially symmetric, we deduce from the last estimate that for any $z \in \Omega_{0,\epsilon}$, there exists $0<\psi_z<\pi$ such that, for all $k \geqslant 2$,

$$\left|\int_{\mathbb{R}} q\big(z+te^{i\psi}\big)\,dt\right| \leqslant A, \quad \psi_z<\psi<\pi+\psi_z-\frac{\pi}{2k}.$$

Therefore

$$\left|\int_{\mathbb{R}} q\big(z+te^{i\psi}\big)\,dt\right| \leqslant A, \quad \psi_z<\psi<\pi+\psi_z.$$

On the other hand,

$$\int_{\mathbb{R}} q\big(z+te^{i\psi}\big)\,dt=\int_{\mathbb{R}} q\big(z-te^{i\psi}\big)\,dt=\int_{\mathbb{R}} q\big(z+te^{i(\psi+\pi)}\big)\,dt.$$

Hence,

$$\left|\int_{\mathbb{R}} q\big(z+te^{i\psi}\big)\,dt\right| \leqslant A, \quad \psi_z<\psi<2\pi+\psi_z.$$

This and the radial symmetry of q imply

$$\big|\mathcal{P}q(\theta,x)\big| \leqslant A, \quad \theta \in \mathbb{S}^1,\ x \in \Omega_\epsilon. \qquad \square$$

Example 2.2. Let us fix $\tilde{q} \in C^{0,\mu}(\overline{\Omega})$ satisfying $\operatorname{supp}(\tilde{q}) \subset \Omega$ and there exists $\delta>0$ such that

$$\|\mathcal{P}\tilde{q}\|_{L^\infty(S\times\Omega_{0,\epsilon})} \geqslant 2\delta.$$

We can take for instance $0 \leqslant \tilde{q} \in C_c^\infty(\Omega)$ satisfying $\tilde{q}=1$ in $B(x_0,\delta) \subset \Omega_{0,\epsilon}$. Indeed we have in this case, for any fixed $\theta \in S$,

$$\mathcal{P}\tilde{q}(\theta,x_0) \geqslant \int_{-\delta}^{\delta} \tilde{q}(x_0+t\theta)\,dt \geqslant 2\delta.$$

Let $B(\tilde{q},r)$ denote the set of functions $q \in C^{0,\mu}(\overline{\Omega})$ satisfying $\operatorname{supp}(q) \subset \Omega$ and

$$\|q-\tilde{q}\|_{L^\infty(\mathbb{R}^d)} \leqslant r.$$

From the definition of the X-ray transform we easily deduce

$$\|\mathcal{P}f\|_{L^\infty(\mathbb{S}^{d-1}\times\mathbb{R}^d)} \leqslant 4R\|f\|_{L^\infty(\mathbb{R}^d)},$$

if $\operatorname{supp}(f) \subset \Omega \subset B(0,R)$.

Therefore any $q \in B(\tilde{q},\rho)$, with $\rho=\frac{\delta}{4R}$, satisfies

$$\|\mathcal{P}q\|_{L^\infty(S\times\Omega_{0,\epsilon})} \geqslant \delta. \tag{2.1}$$

Proposition 2.2. *Let $(q_1,q_2) \in \mathfrak{X}(M,\omega)\times\mathfrak{X}(M,\omega)$ such that $q=q_1-q_2 \in \lambda B(\tilde{q},\rho)$, for some $\lambda \in \mathbb{R}$. Then $(q_1,q_2) \in \mathfrak{X}(M,\omega,S,\Omega_{0,\epsilon})$.*

Proof. Since (1.5) is also satisfied if we replace q by sq, for any $s \in \mathbb{R}$, we may assume that $\lambda=1$. From (2.1), we have

$$\|\mathcal{P}q\|_{L^\infty(\mathbb{S}^{d-1}\times\mathbb{R}^d)} \leqslant 4R\|q_1-q_2\|_{L^\infty(\mathbb{R}^d)} \leqslant 8MR \leqslant \frac{8MR}{\delta}\|\mathcal{P}q\|_{L^\infty(S\times\Omega_{0,\epsilon})}.$$

That is (1.5) holds with $K=\frac{8MR}{\delta}$. $\square$

3. The local DN map for a wave equation

Here we shall use the same notations as in the previous section.

For fixed $T > 0$, we denote $(0, T) \times \Omega$ and $(0, T) \times \Gamma$ respectively by Q and Σ. We consider the following initial-boundary value problem (IBVP in short) for the wave equation.

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma. \end{cases} \tag{3.1}$$

Let

$$H^{1,1}(\Sigma) = L^2\big(0, T; H^1(\Gamma)\big) \cap H^1\big(0, T; L^2(\Gamma)\big).$$

It is proved in Theorem A.3 of [4] that, for any $q \in L^\infty(\Omega)$ and $f \in H^{1,1}(\Sigma)$, the IBVP (3.1) has a unique solution

$$u_q \in C\big([0, T]; H^1(\Omega)\big) \cap C^1\big([0, T]; L^2(\Omega)\big)$$

such that $\partial_\nu u_q \in L^2(\Sigma)$. In addition, for any positive constant M, there exists a positive constant C depending only on Ω, T and M with the property that for all $q \in L^\infty(\Omega)$, $\|q\|_{L^\infty(\Omega)} \leqslant M$, the following estimate holds

$$\|u_q\|_{C([0,T];H^1(\Omega))} + \|u_q\|_{C^1([0,T];L^2(\Omega))} + \|\partial_\nu u_q\|_{L^2(\Sigma)} \leqslant C\|f\|_{H^{1,1}(\Sigma)}.$$

In particular the following operator, usually called the Dirichlet to Neumann map (DN map in short),

$$\begin{aligned} \Lambda_q : H^{1,1}(\Sigma) &\longrightarrow L^2(\Sigma) \\ f &\longmapsto \Lambda_q(f) = \partial_\nu u_q \end{aligned} \tag{3.2}$$

is bounded.

In what follows we assume that T is sufficiently large, at least $T > \operatorname{diam}(\Omega) + 2\epsilon$, where ϵ is the same parameter as in the previous section.

Let γ be an arbitrary closed subset of Γ with nonempty interior. We set $\Sigma_1 = (0, T) \times \Gamma_1$ and

$$H^{1,1}_{\Sigma_1}(\Sigma) = \big\{ f \in H^{1,1}(\Sigma);\ \operatorname{supp}(f) \cap \Sigma_1 = \emptyset \big\}.$$

We introduce the local DN map defined as follows

$$\begin{aligned} \Lambda_q^\sharp : H^{1,1}_{\Sigma_1}(\Sigma) &\longrightarrow L^2(\Sigma_\gamma) \\ f &\longmapsto \Lambda_q^\sharp(f) = \Lambda_q(f)|_{\Sigma_\gamma}, \end{aligned} \tag{3.3}$$

where $\Sigma_\gamma = (0, T) \times \gamma$.

We observe that since Λ_q is bounded, $\Lambda_q^\sharp$ is also bounded when $H^{1,1}_{\Sigma_1}(\Sigma)$ is endowed with the topology of $H^{1,1}(\Sigma)$.

We are now in position to give the statement of the stability estimate for the inverse problem consisting in the determination of q from the local DN map $\Lambda_q^\sharp$.

Theorem 3.1. *There exist $C > 0$, $\delta \in (0, 1)$ and T sufficiently large such that*

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leqslant C\big(\big\|\Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp\big\|^\delta + \big|\log\big\|\Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp\big\|\big|^{-\delta}\big), \tag{3.4}$$

for any $(q_1, q_2) \in \mathfrak{X}(M, \omega, S, \Omega_{0,\epsilon})$.

If in addition $q_1, q_2 \in H^s(\Omega)$, for $s > d/2$ and $\|q_j\|_{H^s(\Omega)} \leqslant M$, $j = 1, 2$, then there exists $\delta' \in (0, 1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\big(\big\|\Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp\big\|^{\delta'} + \big|\log\big\|\Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp\big\|\big|^{-\delta'}\big). \tag{3.5}$$

As an immediate consequence of Theorem 3.1, we have the following uniqueness result.

Corollary 3.1. *There exists T sufficiently large such that if $(q_1, q_2) \in \mathfrak{X}(M, \omega, S, \Omega_{0,\epsilon})$ satisfies $\Lambda_{q_1}^\sharp = \Lambda_{q_2}^\sharp$ then $q_1 = q_2$ in Ω.*

We note that in the special case where Γ_1 is empty, we can choose $S = \mathbb{S}^{d-1}$ and $\Omega_{0,\epsilon} = \Omega_\epsilon$. Therefore (1.5) is automatically satisfied. Let

$$\begin{aligned} \Lambda_q^* : H^{1,1}(\Sigma) &\longrightarrow L^2(\Sigma_\gamma) \\ f &\longmapsto \Lambda_q^*(f) = \Lambda_q|_{\Sigma_\gamma}. \end{aligned}$$

Then we have the following statement

Theorem 3.2. *There exist $C>0$, $\delta\in(0,1)$ and T sufficiently large such that*

$$\|q_1-q_2\|_{H^{-1/2}(\Omega)}\leqslant C\big(\big\|\Lambda^*_{q_1}-\Lambda^*_{q_2}\big\|^{\delta}+\big|\log\big\|\Lambda^*_{q_1}-\Lambda^*_{q_2}\big\|\big|^{-\delta}\big),$$

for any $q_1,q_2\in\mathfrak{X}(M,\omega)$.

If in addition $q_1,q_2\in H^s(\Omega)$, for $s>d/2$ and $\|q_j\|_{H^s(\Omega)}\leqslant M$, $j=1,2$, then there exists $\delta'\in(0,1)$ such that

$$\|q_1-q_2\|_{L^\infty(\Omega)}\leqslant C\big(\big\|\Lambda^*_{q_1}-\Lambda^*_{q_2}\big\|^{\delta'}+\big|\log\big\|\Lambda^*_{q_1}-\Lambda^*_{q_2}\big\|\big|^{-\delta'}\big).$$

This last theorem was proved in [4] and Theorem 3.1 sharpens the result of [4] under the extra condition $(q_1,q_2)\in\mathfrak{X}(M,\omega,S,\Omega_{0,\epsilon})$.

Our proof of Theorem 3.1 follows the same lines to that of Theorem 1.1 in [4]. The main difference is that we use in place of Lemma 3.2 of [4] the following proposition.

Proposition 3.1. *There exist four constants $C>0$, $A>0$, $\delta>0$ and λ_0 such that for all $\theta\in\mathbb{S}^{d-1}$,*

$$\bigg|\int_{\mathbb{R}}q(y+s\theta)\,ds\bigg|\leqslant\frac{C}{\lambda^\delta}+Ce^{A\lambda}\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|,\quad \text{a.e. } y\in\mathbb{R}^d,$$

for any $\lambda\geqslant\lambda_0$.

We give the proof of this proposition in Appendix A.

Proof of Theorem 3.1. Let $(q_1,q_2)\in\mathfrak{X}(M,\omega,S,\Omega_{0,\epsilon})$ and $q=q_1-q_2$ extended outside of Ω by zero. From Proposition 3.1 we have

$$\big|\mathcal{P}(q)(x,\theta)\big|=\bigg|\int_{\mathbb{R}}q(x+s\theta)\,ds\bigg|\leqslant C\bigg(\frac{1}{\lambda^\delta}+e^{\mu\lambda}\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|\bigg),\quad \text{a.e. } x\in\mathbb{R}^d.$$

We choose $R>0$ such that $\Omega\subset B(0,R)=\{x\in\mathbb{R}^n;\ |x|<R\}$. Then

$$\big\|\mathcal{P}(q)\big\|^2_{L^2(\mathcal{T})}:=\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp\cap B(0,R)}\big|\mathcal{P}(q)(\theta,y)\big|^2\,dy\,d\theta\leqslant C\bigg(\frac{1}{\lambda^\delta}+e^{\mu\lambda}\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|\bigg),$$

where $\mathcal{T}=\{(\theta,y);\ \theta\in\mathbb{S}^{d-1},\ y\in\theta^\perp\}$ is the tangent bundle.

We recall the following well-known estimate for the X-ray transform (e.g. [17])

$$\|q\|_{H^{-\frac12}(\Omega)}\leqslant C\big\|\mathcal{P}(q)\big\|^2_{L^2(\mathcal{T})}.$$

Therefore

$$\|q\|_{H^{-\frac12}(\Omega)}\leqslant C\bigg(\frac{1}{\lambda^\delta}+e^{\mu\lambda}\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|\bigg).\tag{3.6}$$

This estimate is valid if $\lambda\geqslant\lambda_0$. Hence there exists $\epsilon_0<<1$ such that if $\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\|<\epsilon_0$ and

$$\lambda=\frac{1-\delta}{\mu}\big|\log\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|\big|$$

we have $\lambda\geqslant\lambda_0$. From (3.6) we obtain

$$\|q\|_{H^{-1/2}(\Omega)}\leqslant C\big(\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|^{\delta}+C'\big|\log\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|\big|^{-\delta}\big)\tag{3.7}$$

when $\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\|<\epsilon_0$. On the other hand, $\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\|\geqslant\epsilon_0$ readily implies

$$\|q\|_{H^{-1/2}(\Omega)}\leqslant C\|q\|_{L^\infty(\Omega)}\leqslant\frac{2CM}{\epsilon_0^\delta}\epsilon_0^\delta\leqslant C'\big\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\big\|^{\delta}.$$

Thus (3.7) holds also in the present case.

The second estimate is a consequence of the Sobolev imbedding theorem and an interpolation inequality. Let $\eta>0$ such that $s=d/2+2\eta$. Then we have

$$\|q\|_{L^\infty(\Omega)}\leqslant C\|q\|_{H^{s-\eta}(\Omega)}\leqslant C\|q\|^{\beta}_{H^{-1/2}(\Omega)}\|q\|^{1-\beta}_{H^s(\Omega)}\leqslant C\|q\|^{\beta}_{H^{-1/2}(\Omega)},$$

where

$$\beta = \frac{s-d/2}{2s+1} < 1.$$

The conclusion follows from (3.7). $\square$

For proving the stability estimate for our inverse spectral problem, we need to restrict the operator $\Lambda_q^\sharp$ to the following subspace

$$\mathscr{F}_{\Sigma_\gamma} = \left\{ g \in H^{2d+4}\left(0,T; H^{\frac{3}{2}}(\Gamma)\right);\ \partial_t^j g(0,\cdot) = 0,\ 0 \leqslant j \leqslant 2d+3, \text{ and } \operatorname{supp}(g) \subset \Sigma_\gamma \right\}.$$

This subspace is endowed with the topology of $H^{2d+4}(0,T;H^{\frac{3}{2}}(\Gamma))$. We will see below that this operator, denoted by $\tilde{\Lambda}_q$, defines a bounded operator from $\mathscr{F}_{\Sigma_\gamma}$ into $L^2(0,T;H^\alpha(\gamma))$ for any α, $0 \leqslant \alpha \leqslant 1/2$.

We have the following variant of Theorem 3.1, where $\|\cdot\|_\alpha$ denotes the norm in $\mathcal{B}(\mathscr{F}_{\Sigma_\gamma}, L^2(0,T;H^\alpha(\gamma)))$.

Theorem 3.3. *There exist $C > 0$, $\delta \in (0,1)$ and T sufficiently large such that*

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leqslant C\left(\|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha^\delta + \left|\log \|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha\right|^{-\delta}\right), \tag{3.8}$$

for any $(q_1,q_2) \in \mathfrak{X}(M,\omega,S,\Omega_{0,\epsilon})$.

If in addition $q_1, q_2 \in H^s(\Omega)$, for $s > d/2$ and $\|q_j\|_{H^s(\Omega)} \leqslant M$, $j = 1,2$, then there exists $\delta' \in (0,1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\left(\|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha^{\delta'} + \left|\log \|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha\right|^{-\delta'}\right). \tag{3.9}$$

Proof. We use the same notations as in the proof of Lemma A.3 in Appendix A. From $(A.13)$. Especially, f_σ is given in the proof of Lemma A.3. We can easily deduce

$$\begin{aligned}\|\partial_\nu u\|_{L^2(\Sigma_\gamma)} &= \left\|\tilde{\Lambda}_{q_1}(f_\sigma) - \tilde{\Lambda}_{q_2}(f_\sigma)\right\|_{L^2(\Sigma_\gamma)} \leqslant C\left\|\tilde{\Lambda}_{q_1}(f_\sigma) - \tilde{\Lambda}_{q_2}(f_\sigma)\right\|_{L^2(0,T;H^\alpha(\gamma))} \\ &\leqslant C\|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha \|f_\sigma\|_{H^{2d+4}(0,T;H^{3/2}(\gamma))}.\end{aligned}$$

On the other hand, one can easily establish the following estimate (proved in p. 79 of [8])

$$\|f_\sigma\|_{H^{2d+4}(0,T;H^{3/2}(\gamma))} \leqslant C\sigma^{2d+4}.$$

Therefore

$$\|\partial_\nu u\|_{L^2(\Sigma_\gamma)} \leqslant C\sigma^{2d+4}\|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha.$$

In this case, in place of $(A.14)$ we have

$$\left|\int_Q q(x)\Phi^2(x+t\theta)\,dx\,dt\right| \leqslant C\left(\frac{1}{\sigma} + \frac{\sigma}{\sqrt{\lambda}} + \sigma^{2d+4}e^{\mu\lambda}\|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)}.$$

As in the proof of Lemma A.3, by taking $\sigma = \lambda^{1/4}$, we find

$$\left|\int_Q q(x)\Phi^2(x+t\theta)\,dx\,dt\right| \leqslant C\left(\frac{1}{\lambda^{1/4}} + e^{C\lambda}\|\tilde{\Lambda}_{q_1} - \tilde{\Lambda}_{q_2}\|_\alpha\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)}.$$

That is we have Lemma A.3 with $\tilde{\Lambda}_{q_i}$ replaced by $\Lambda_{q_i}^\sharp$.

The rest of the proof is similar to that of Theorem 3.1. $\square$

Here we refer to results on DN maps for hyperbolic equations. The uniqueness in determining q from the full DN map Λ_q was established by Rakesh and Symes [19], under the usual geometric condition insuring that the length of the time interval is larger than the diameter of the space domain Ω. A sharp uniqueness result was proved in [14] by the so-called boundary control method. The case of piecewise potential q was considered later by Rakesh [18]. He proves the uniqueness and a stability estimate from the values of $\Lambda_q(f)$, f suitably chosen in a finite subset. The uniqueness and a Hölder stability estimate were established by Isakov and Sun [13] for a partial DN map by tools from the integral geometry.

Based on a method using global Carleman estimates, Bellassoued, Jellali and Yamamoto [5] prove a Lipschitz stability estimate when the potential is restricted to a finite dimensional subspace. In [6], by the same method they prove a log–log type stability estimate in the case where the data is a partial DN map. The authors [4] proved a log type stability estimate for the partial DN map $\Lambda_q^\sharp$, under the assumption that the potential is known near the boundary. Related results for hyperbolic inverse problems were proved for example by [2,3,7,9,12,15,11,20] and many others.

4. Proof of the stability for the spectral problem

Our proof of Theorem 1.1 is similar to that of Theorem 1.1 in [4]. For sake of completeness, we shall give all details. We first define an elliptic DN map. Let $q \in L^\infty(\Omega)$, $\sigma(A_q) = \{\lambda_{k,q}\}$ be the spectrum of A_q and $\rho(A_q) = \mathbb{C} \setminus \sigma(A_q)$ be the resolvent set of A_q. From well-known results (e.g. [16]), for any $\lambda \in \rho(A_q)$ and $f \in H^{3/2}(\Gamma)$, the nonhomogeneous BVP

$$\begin{cases} -\Delta u + qu - \lambda u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Gamma, \end{cases}$$

has a unique solution $u_{q,f} \in H^2(\Omega)$ and the DN map

$$\Pi_q : f \to \partial_\nu u_{q,f|\gamma}$$

define a bounded operator from $H_\gamma^{3/2}(\Gamma) = \{f \in H^{3/2}(\Gamma);\ \operatorname{supp}(f) \subset \gamma\}$ into $H^{1/2}(\gamma)$.

We shall need in the sequel the following three lemmas. Their proof can be found in [1] or can be deduced easily from the results in this reference (see also [8]). Hereafter, $0 \leqslant \alpha < \frac{1}{2}$ is fixed.

Lemma 4.1. *Let $q \in L^\infty(\Omega)$. Then for any $m > \frac{d}{2}$, $f \in H_\gamma^{\frac{3}{2}}(\Gamma)$ and $\lambda \in \rho(A_q)$,*

$$\frac{d^m}{d\lambda^m} \Pi_q(\lambda) f = -m! \sum_{k \geqslant 1} \frac{1}{(\lambda_{k,q} - \lambda)^{m+1}} \langle f, \partial_\nu \varphi_{k,q} \rangle \partial_\nu \varphi_{k,q|\gamma}.$$

Here

$$\langle f, \partial_\nu \varphi_{k,q} \rangle = \int_\Gamma f \partial_\nu \varphi_{k,q}\, d\sigma.$$

Lemma 4.2. *Let n be a non-negative integer and $q_1, q_2 \in L^\infty(\Omega)$ satisfying $0 \leqslant q_1, q_2 \leqslant M$, for some positive constant M. Then there exists a positive constant C, depending only on Ω and M, such that*

$$\left\| \frac{d^j}{d\lambda^j} \left[\Pi_{q_1}(\lambda) - \Pi_{q_2}(\lambda) \right] \right\|_\alpha \leqslant \frac{C}{|\lambda|^{j + \frac{1-2\alpha}{4}}}, \quad \lambda \leqslant 0 \text{ and } 0 \leqslant j \leqslant n,$$

where $\|\cdot\|_\alpha$ denotes the norm in $\mathcal{L}(H^{\frac{3}{2}}(\gamma), H^\alpha(\gamma))$.

Lemma 4.3. *Let*

$$\mathscr{F}_{\Sigma_\gamma} = \left\{ g \in H^{2d+4}\big(0, T; H^{\frac{3}{2}}(\Gamma)\big);\ \partial_t^j g(0, \cdot) = 0,\ 0 \leqslant j \leqslant 2d+3,\ \text{and } \operatorname{supp}(g) \subset \Sigma_\gamma \right\},$$

where $\Sigma_\gamma = (0, T) \times \gamma$. Then for each $f \in \mathscr{F}_{\Sigma_\gamma}$,

$$\tilde{\Lambda}_q f = \sum_{j=0}^{d+1} \left[\frac{d^j}{d\lambda^j} \Pi_q(\lambda) \right]_{|\lambda=0} \big(-\partial_t^2 f\big) + \mathscr{R}_q f_{|\gamma}, \tag{4.1}$$

where $\tilde{\Lambda}_q$ is as in Theorem 2.3 and

$$\mathscr{R}_q f = \sum_{k \geqslant 1} \frac{1}{\lambda_{q,k}^{d+\frac{5}{2}}} \partial_\nu \varphi_{q,k} \int_0^t \sin \sqrt{\lambda_{q,k}}(t-s)\, ds \big\langle -\partial_s^{2(d+2)} f(\cdot, s), \partial_\nu \varphi_{q,k} \big\rangle.$$

According to this lemma, we note that $\tilde{\Lambda}_q$, with $q \in L^\infty(\Omega)$, defines a bounded operator from $\mathscr{F}_{\Sigma_\gamma}$ into $H^\alpha(\gamma)$.

Let $f \in H_\gamma^{3/2}(\Gamma)$. We set $F(\lambda) = (\Pi_{q_1}(\lambda) - \Pi_{q_2}(\lambda)) f$. Then from Taylor's formula, for $1 \leqslant j \leqslant d$, we derive

$$F^{(j)}(0) = \sum_{p=j}^{d} \frac{(-\lambda)^{p-j}}{(p-j)!} F^{(p)}(\lambda) + \int_\lambda^0 \frac{(-\tau)^{d-j}}{(d-j)!} F^{(d+1)}(\tau)\, d\tau.$$

We recall that

$$\eta = d_\infty(\lambda_{q_1}, \lambda_{q_2}) + \|\partial_\nu \varphi_{q_1} - \partial_\nu \varphi_{q_2}\|_{\ell^1(H^{\frac{1}{2}}(\gamma), w)}.$$

Lemma 4.4. *The following estimate holds*

$$\left\| F^{(d+1)}(\lambda)\right\|_{\alpha} \leqslant C\eta, \tag{4.2}$$

where C is a positive constant depending on M and Ω.

Proof. It follows from Lemma 4.1

$$F^{(d+1)}(\lambda) = -(d+1)!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}}\langle f, \partial_\nu\varphi_{k,q_1}\rangle\partial_\nu\varphi_{k,q_1|\gamma} + (d+1)!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\langle f, \partial_\nu\varphi_{k,q_2}\rangle\partial_\nu\varphi_{k,q_2|\gamma}.$$

We split $F^{(d+1)}(\lambda)$ into three terms $F^{(d+1)}(\lambda) = I_1(\lambda)+I_2(\lambda)+I_3(\lambda)$, where

$$I_1(\lambda) = -(d+1)!\sum_{k\geqslant 1}\left[\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}} - \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\right]\langle f, \partial_\nu\varphi_{k,q_1}\rangle\partial_\nu\varphi_{k,q_1|\gamma},$$

$$I_2(\lambda) = -(d+1)!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\langle f, \partial_\nu\varphi_{k,q_1}-\partial_\nu\varphi_{k,q_2}\rangle\partial_\nu\varphi_{k,q_1|\gamma},$$

$$I_3(\lambda) = -(d+1)!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\langle f, \partial_\nu\varphi_{k,q_2}\rangle[\partial_\nu\varphi_{k,q_1}-\partial_\nu\varphi_{k,q_2}]_{|\gamma}.$$

For I_1, we have

$$\left\| I_1(\lambda)\right\|_{H^{\frac{1}{2}}(\Gamma)} \leqslant (d+1)!\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\left|\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}} - \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\right|\|\partial_\nu\varphi_{k,q_2}\|^2_{H^{\frac{1}{2}}(\gamma)}.$$

Moreover

$$\left|\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}} - \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\right| \leqslant C\max\left(\frac{1}{\lambda_{k,q_1}^{d+3}}, \frac{1}{\lambda_{k,q_2}^{d+3}}\right)|\lambda_{k,q_1}-\lambda_{k,q_2}| \leqslant \frac{C}{k^{\frac{2(d+3)}{d}}}|\lambda_{k,q_1}-\lambda_{k,q_2}|,$$

where we used estimate (1.1). On the other hand, since (see (1.2))

$$\|\partial_\nu\varphi_{k,q_2}\|^2_{H^{\frac{1}{2}}(\gamma)} \leqslant Ck^{\frac{4}{d}},$$

we obtain

$$\left\| I_1(\lambda)\right\|_{H^{\frac{1}{2}}(\gamma)} \leqslant C\|f\|_{L^2(\Gamma)}d_\infty(\lambda_{q_1},\lambda_{q_2})\sum_{k\geqslant 1}\frac{1}{k^{\frac{2(d+1)}{d}}} \leqslant C\|f\|_{L^2(\Gamma)}d_\infty(\lambda_{q_1},\lambda_{q_2}). \tag{4.3}$$

We similarly proceed to prove

$$\begin{aligned}\left\| I_2(\lambda)\right\|_{H^{\frac{1}{2}}(\gamma)} + \left\| I_3(\lambda)\right\|_{H^{\frac{1}{2}}(\gamma)} &\leqslant C\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\left(\frac{\lambda_{k,q_1}+\lambda_{k,q_2}}{\lambda_{k,q_2}^{d+2}}\right)\|\partial_\nu\varphi_{k,q_1}-\partial_\nu\varphi_{k,q_2}\|_{H^{\frac{1}{2}}(\gamma)}\\ &\leqslant C\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\frac{1}{k^{\frac{2(d+1)}{d}}}\|\partial_\nu\varphi_{k,q_2}-\partial_\nu\varphi_{k,q_1}\|_{H^{\frac{1}{2}}(\gamma)}\\ &\leqslant C\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\frac{1}{k^{\frac{2\zeta}{d}}}\|\partial_\nu\varphi_{k,q_2}-\partial_\nu\varphi_{k,q_1}\|_{H^{\frac{1}{2}}(\gamma)}.\end{aligned}$$

Therefore

$$\|I_2\|_{H^{\frac{1}{2}}(\gamma)} + \|I_3\|_{H^{\frac{1}{2}}(\gamma)} \leqslant C\|f\|_{L^2(\Gamma)}\|\partial_\nu\varphi_{q_1}-\partial_\nu\varphi_{q_2}\|_{\ell^1(H^{\frac{1}{2}}(\gamma),w)}. \tag{4.4}$$

The conclusion follows then from a combination of (4.3) and (4.4). □

Proof of Theorem 1.1. From Lemma 4.2, we obtain the following estimate

$$\left\|F^{(j)}(0)\right\|_\alpha \leqslant C\left(|\lambda|^{-\frac{1-2\alpha}{4}}+|\lambda|^{d-j+1}\eta\right),$$

and then

$$\left\|F^{(j)}(0)\right\|_\alpha \leqslant C\left(|\lambda|^{-\frac{1-2\alpha}{4}}+|\lambda|^{d+1}\eta\right), \quad \text{if } |\lambda| \geqslant 1.$$

In particular

$$\left\|F^{(j)}(0)\right\|_\alpha \leqslant C \min_{\rho\geqslant 1}\left(\rho^{-\frac{1-2\alpha}{4}}+\rho^{d+1}\eta\right).$$

By choosing $\rho \geqslant 1$ giving the minimum, we deduce

$$\left\|F^{(j)}(0)\right\|_\alpha \leqslant C\eta^\theta, \tag{4.5}$$

with $\theta = 1-\frac{4(d+1)}{(1-2\alpha)+4(d+1)}$. Let $\mathscr{R}_q$ be defined as in Lemma 4.3. We can proceed as in the proof of the preceding lemma to prove

$$\|\mathscr{R}_{q_1}-\mathscr{R}_{q_2}\|_\alpha \leqslant C\eta. \tag{4.6}$$

From identity (4.1), estimates (4.5) and (4.6), we deduce

$$\|\tilde{\Lambda}_{q_1}-\tilde{\Lambda}_{q_2}\|_\alpha \leqslant C\left(\eta+\eta^\theta\right).$$

This and the estimates in Theorem 3.3 lead to the desired result. □

Appendix A. Proof of Proposition 3.1

We first recall a result on the existence of geometric optic solutions, which is due to Rakesh and Symes [19].

Lemma A.1. *Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\theta \in \mathbb{S}^{d-1}$, $0 \neq \sigma \in \mathbb{R}$. Then the equation*

$$\partial_t^2 u - \Delta u + q(x)u = 0, \quad \text{in } Q$$

has a solution of the form

$$u(t,x) = \Phi(x+t\theta)e^{i\sigma(x\cdot\theta+t)} + \psi_q(t,x;\sigma), \tag{A.1}$$

where $\psi_q(t,x;\sigma)$ satisfies

$$\psi_q(t,x;\sigma) = 0, \quad \forall (t,x) \in \Sigma,$$
$$\psi_q(\tau,x;\sigma) = 0, \quad x \in \Omega,\ \tau = 0 \text{ or } T$$

and

$$|\sigma|\left\|\psi_q(\cdot,\cdot;\sigma)\right\|_{L^2(Q)} + \left\|\nabla\psi_q(\cdot,\cdot;\sigma)\right\|_{L^2(Q)} \leqslant C\|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{A.2}$$

Here the constant C depends only on T, Ω and M (that is, C does not depend on Φ and σ).

We refer to [19] for the proof.

We shall use the following notations. We choose $\varrho > 0$ such that

$$\omega(8\varrho) = \left\{x \in \Omega;\ \mathrm{dist}(x,\Gamma) \leqslant 8\varrho\right\} \subset \omega$$

and, for $\tau > 0$, we set

$$\omega_\tau = (0,\tau)\times\omega, \qquad \omega_\tau(\varrho) = (0,\tau)\times\omega(\varrho).$$

The main ingredient in the proof of Proposition 3.1 is a stability estimate for the unique continuation near the boundary for a wave equation from a lateral boundary data on Σ_γ.

Lemma A.2. *(See [4, Lemma 2.2].) Let $q \in \mathfrak{X}(M,\omega)$ and T sufficiently large. Let $w \in H^2(Q)$ be a solution of the following boundary value problem*

$$\begin{cases} (\partial_t^2 - \Delta + q(x))w = F, & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \end{cases}$$

where $F \in L^2(Q)$. Then there exist positive constants C, $T' > T/3$, μ and λ_0 such that the following estimate holds

$$\|w\|_{H^1(\omega_{T'}(2\varrho))} \leqslant \frac{C}{\sqrt{\lambda}}\|w\|_{H^2(Q)} + e^{\mu\lambda}\big(\|F\|_{L^2(\omega_T)} + \|\partial_\nu w\|_{L^2(\Sigma_\gamma)}\big),$$

for any $\lambda > \lambda_0$. Here the constant C depends on Ω, ω, T, M and it is independent of w and q.

To $\Phi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$ and $\theta \in \mathbb{S}^{d-1}$ we associate $\tilde{\Phi}_\theta$ defined by

$$\tilde{\Phi}_\theta(t,x) = \Phi(x + t\theta), \quad x \in \mathbb{R}^d,\ t \in \mathbb{R},$$

where Φ is extended outside Ω_ε by 0.

We recall that $\Sigma_1 = (0,T) \times \Gamma_1$.

Lemma A.3. *Let $q_1, q_2 \in \mathfrak{X}(M,\omega)$ and let q be equal to $q_1 - q_2$ extended outside Ω by 0. There exist $T' > 0$ sufficiently large, $A > 0$ and $C > 0$ such that for any $\theta \in S$ and any $\Phi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$ such that* $\operatorname{supp}(\tilde{\Phi}_\theta) \cap \Sigma_1 = \emptyset$,

$$\left|\int_0^{T'}\int_\Omega \Phi^2(x) q(x - s\theta)\,dx\,ds\right| \leqslant C\left(\frac{1}{\lambda^{1/4}} + e^{A\lambda}\big\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\big\|\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)},$$

for sufficiently large $\lambda > 0$.

Proof. Let T' be as in Lemma A.2. It follows from Lemma A.1 that if σ is sufficiently large, then the initial value problem

$$\big(\partial_t^2 - \Delta + q_2(x)\big)u = 0 \quad \text{in } Q' = \big(0,T'\big) \times \Omega, \qquad u(0,\cdot) = \partial_t u(0,\cdot) = 0 \quad \text{in } \Omega$$

has a solution u_2 of the form

$$u_2(t,x) = \Phi(x + t\theta)e^{i\sigma(x.\theta + t)} + \psi_{q_2}(t,x;\sigma), \tag{A.3}$$

where ψ_{q_2} satisfies

$$\psi_{q_2}(0,x;\sigma) = \partial_t\psi_{q_2}(0,x;\sigma) = 0, \qquad \psi_{q_2}(t,x;\sigma) = 0 \quad \text{on } \Sigma' = \big(0,T'\big) \times \Gamma \tag{A.4}$$

and

$$|\sigma|\big\|\psi_{q_2}(\cdot,\cdot;\sigma)\big\|_{L^2(Q)} + \big\|\nabla\psi_{q_2}(\cdot,\cdot;\sigma)\big\|_{L^2(Q)} \leqslant C\|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{A.5}$$

Let $f_\sigma := u_{2|\Sigma'} = \Phi(x + t\theta)e^{i\sigma(x\cdot\theta + t)}$ and let u_1 be the solution of the IBVP

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + q_1(x)u_1 = 0, & \text{in } Q', \\ u_1(0,x) = \partial_t u_1(0,x) = 0, & \text{in } \Omega, \\ u_1 = u_2 := f_\sigma, & \text{on } \Sigma'. \end{cases}$$

We set $u = u_1 - u_2$ and $q(x) = q_2(x) - q_1(x)$. Then we easily prove

$$\begin{cases} \partial_t^2 u - \Delta u + q_1(x)u = q(x)u_2, & \text{in } Q', \\ u(0,x) = \partial_t u(0,x) = 0, & \text{in } \Omega, \\ u = 0, & \text{in } \Sigma'. \end{cases}$$

We introduce a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$ satisfying $0 \leqslant \chi \leqslant 1$ and

$$\begin{cases} 0, & x \in \omega(\varrho), \\ 1, & x \in \Omega \setminus \omega(2\varrho). \end{cases}$$

If $w(t,x) = \chi(x)u(t,x)$ then we see that w is the solution of the following IBVP

$$\begin{cases} \partial_t^2 w - \Delta w + q_1(x)w = q(x)u_2 + [\Delta,\chi]u, & \text{in } Q', \\ w(0,\cdot) = \partial_t w(0,\cdot) = 0, & \text{in } \Omega, \\ w = 0, & \text{on } \Sigma', \end{cases}$$

where we used $\chi(x)q(x) = q(x)$ because $q(x) = 0$ in ω. Therefore, by Green's formula, for an arbitrary $v \in H^1(Q)$ such that $v(T,\cdot) = \partial_t v(T,\cdot) = 0$, we obtain

$$\int_{Q'} \big(\partial_t^2 - \Delta + q_1(x)\big) w v \, dx\, dt = \int_{Q'} q(x) u_2(t,x) v \, dx\, dt + \int_{Q'} [\Delta, \chi] u v \, dt\, dx$$
$$= \int_{Q'} w \big(\partial_t^2 - \Delta + q_1(x)\big) v \, dx\, dt. \tag{A.6}$$

On the other hand, for sufficiently large σ, Lemma A.1 guarantees the existence of exponentially growing solutions v to the backward wave equation

$$\big(\partial_t^2 - \Delta + q_1(x)\big) v = 0, \quad \text{in } Q', \qquad v\big(T', \cdot\big) = \partial_t v\big(T', \cdot\big) = 0, \quad \text{in } \Omega$$

of the form

$$v(t,x) = \Phi(x + t\theta) e^{-i\sigma(x\cdot\theta + t)} + \psi_{q_1}(t,x,\sigma), \tag{A.7}$$

where ψ_{q_1} satisfies

$$|\sigma| \big\| \psi_{q_1}(\cdot,\cdot;\sigma) \big\|_{L^2(Q')} + \big\| \nabla \psi_{q_1}(\cdot,\cdot;\sigma) \big\|_{L^2(Q')} \leqslant C \|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{A.8}$$

It follows from $(A.3)$, $(A.6)$ and $(A.7)$

$$\int_{Q'} q(x) \Phi^2(x + t\theta)\, dx\, dt + \int_{Q'} q(x) \Phi(x + t\theta) \big(e^{i\sigma(x\cdot\theta + t)} \psi_{q_1}(t,x;\sigma) + e^{-i\sigma(x\cdot\theta + t)} \psi_{q_2}(t,x;\sigma) \big)\, dx\, dt$$
$$+ \int_{Q'} q(x) \psi_{q_1}(t,x;\sigma) \psi_{q_2}(t,x;\sigma)\, dx\, dt = \int_{Q'} [\Delta, \chi] u(t,x) v(t,x)\, dt\, dx. \tag{A.9}$$

Now (A.5) and (A.8) imply

$$\left| \int_{Q'} q(x) \Phi(x + t\theta) \big(e^{i\sigma(x\cdot\theta + t)} \psi_{q_1}(t,x;\sigma) + e^{-i\sigma(x\cdot\theta + t)} \psi_{q_2}(t,x;\sigma) \big)\, dx\, dt \right| \leqslant \frac{C}{|\sigma|} \|\Phi\|^2_{H^3(\mathbb{R}^d)}$$

and

$$\left| \int_{Q'} q(x) \psi_{q_1}(t,x;\sigma) \psi_{q_2}(t,x;\sigma)\, dx\, dt \right| \leqslant \frac{C}{\sigma^2} \|\Phi\|^2_{H^3(\mathbb{R}^d)}.$$

Furthermore

$$\left| \int_{Q'} [\Delta, \chi] u(t,x) v(t,x)\, dx\, dt \right| \leqslant \|u\|_{H^1(\omega_{T'}(2\varrho))} \|v\|_{L^2(Q')} \leqslant C \|\Phi\|_{H^3(\mathbb{R}^d)} \|u\|_{H^1(\omega_{T'}(2\varrho))}. \tag{A.10}$$

Hence, we get from (A.9)

$$\left| \int_{Q'} q(x) \Phi^2(x + t\theta)\, dx\, dt \right| \leqslant \frac{C}{\sigma} \|\Phi\|^2_{H^3(\mathbb{R}^d)} + C \|u\|_{H^1(\omega_{T'}(2\varrho))} \|\Phi\|_{H^3(\mathbb{R}^d)}.$$

This and Lemma A.2 yield

$$\left| \int_{Q'} q(x) \Phi^2(x + t\theta)\, dx\, dt \right| \leqslant \frac{C}{\sigma} \|\Phi\|^2_{H^3(\mathbb{R}^d)} + C \left(\frac{1}{\sqrt{\lambda}} \|u\|_{H^2(Q')} + e^{\mu\lambda} \|\partial_\nu u\|_{L^2(\Sigma_\gamma)} \right) \|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{A.11}$$

By the energy estimate we have

$$\|u\|_{H^2(Q')} \leqslant C\sigma \|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{A.12}$$

On the other hand,

$$\|\partial_\nu u\|_{L^2(\Sigma_\gamma)} = \big\| \Lambda^\sharp_{q_1}(f_\sigma) - \Lambda^\sharp_{q_2}(f_\sigma) \big\|_{L^2(\Sigma_\gamma)} \leqslant C \big\| \Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2} \big\| \|f_\sigma\|_{H^{1,1}(\Sigma)}$$
$$\leqslant C\sigma^2 \big\| \Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2} \big\| \|\Phi\|_{H^3(\mathbb{R}^d)}, \tag{A.13}$$

where we used the fact that $f_\sigma \in H^{1,1}_{\Sigma_1}(\Sigma')$. Now $(A.11)$, $(A.12)$ and $(A.13)$ imply

$$\left|\int_{Q'} q(x)\Phi^2(x+t\theta)\,dx\,dt\right| \leqslant C\left(\frac{1}{\sigma}+\frac{\sigma}{\sqrt{\lambda}}+\sigma^2 e^{\mu\lambda}\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)}. \tag{A.14}$$

Choosing $\sigma=\lambda^{1/4}$, we find

$$\left|\int_0^{T'}\int_\Omega q(x)\Phi^2(x+t\theta)\,dx\,dt\right| \leqslant C\left(\frac{1}{\lambda^{1/4}}+e^{A\lambda}\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)}.$$

By using the substitution $x\to x+s\theta$, we obtain the desired estimate. $\square$

Proof of Proposition 3.1. We fix $\theta\in S$ and $\varphi\in\mathcal{C}^\infty_0(\mathbb{R}^d)$ a non-negative function supported in the unit ball with $\|\varphi\|_{L^2(\mathbb{R}^d)}=1$. We define

$$\Phi_\kappa(x)=\kappa^{-d/2}\varphi\left(\frac{x-y}{\kappa}\right),$$

where $y\in\Omega_{0,\varepsilon}$ and $\kappa>0$ is small enough.

If

$$h(x,\theta)=\int_0^{T'} q(x-t\theta)\,dt,$$

then

$$\left|h(y,\theta)\right|=\left|\int_{\mathbb{R}^d}\Phi^2_\kappa(x)h(y,\theta)\,dx\right|\leqslant\left|\int_{\mathbb{R}^d}\Phi^2_\kappa(x)h(x,\theta)\,dx\right|+\left|\int_{\mathbb{R}^d}\Phi^2_\kappa(x)\big(h(y,\theta)-h(x,\theta)\big)\,dx\right|.$$

Since

$$\left|h(y,\theta)-h(x,\theta)\right|\leqslant\begin{cases} C|x-y|^\mu & \text{if } q_j\in\mathcal{C}^{0,\mu}(\mathbb{R}^d),\\ C|x-y|^{\mu'} & \text{if } q_j\in H^s(\mathbb{R}^d),\ 0<\mu'<\min(s-d/2,1),\end{cases}$$

we obtain

$$\left|h(y,\theta)\right|\leqslant C\left(\frac{1}{\lambda^{1/4}}+e^{\mu\lambda}\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|\right)\|\Phi_\kappa\|^2_{H^3(\mathbb{R}^d)}+C\int_{\mathbb{R}^d}\big(|x-y|^\mu+|x-y|^{\mu'}\big)\Phi^2_\kappa(x)\,dx.$$

Moreover

$$\|\Phi_\kappa\|_{H^3(\mathbb{R}^d)}\leqslant C\kappa^{-3}$$

and, for $\mu_0=\min(\mu,\mu')$,

$$\int_{\mathbb{R}^d}\big(|x-y|^\mu+|x-y|^{\mu'}\big)\Phi^2_\kappa(x)\,dx\leqslant C\kappa^{\mu_0}.$$

Then by Lemma A.3 we have for all $\theta\in S$

$$\left|\int_0^{T'} q(y-t\theta)\,dt\right|\leqslant\frac{C}{\lambda^{1/4}}\kappa^{-6}+C\kappa^{-6}e^{\mu\lambda}\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|+C\kappa^{\mu_0},\quad \text{a.e. } y\in\Omega_{0,\varepsilon}.$$

We select κ satisfying

$$\kappa^{\mu_0}=\frac{1}{\lambda^{1/4}}\kappa^{-6}.$$

From the last estimate we derive that there exist $\delta>0$ and $B>0$ such that

$$\left|\int_{-T'}^{T'} q(y+t\theta)\,dt\right|\leqslant\frac{C}{\lambda^\delta}+Ce^{B\lambda}\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|,\quad \text{a.e. } y\in\Omega_{0,\varepsilon}.$$

Using that $T' > \mathrm{Diam}(\Omega)$ and $\mathrm{supp}(q) \subset \Omega$, we obtain

$$\left|\mathcal{P}(q)(\theta,x)\right| = \left|\int_{\mathbb{R}} q(x+t\theta)\,dt\right| \leqslant \frac{C}{\lambda^{\delta}} + Ce^{B\lambda}\left\|\Lambda_{q_1}^{\sharp} - \Lambda_{q_2}^{\sharp}\right\|, \quad \theta \in S, \text{ a.e. } y \in \Omega_{0,\varepsilon}.$$

In view of assumption (1.5), the last estimate implies

$$\left|\mathcal{P}(q)(\theta,x)\right| \leqslant \frac{C}{\lambda^{\delta}} + Ce^{B\lambda}\left\|\Lambda_{q_1}^{\sharp} - \Lambda_{q_2}^{\sharp}\right\|, \quad \theta \in \mathbb{S}^{d-1}, \text{ a.e. } y \in \Omega_{\varepsilon}.$$

Therefore

$$\left|\mathcal{P}(q)(\theta,x)\right| = \left|\int_{\mathbb{R}} q(x+t\theta)\,dt\right| \leqslant \frac{C}{\lambda^{\delta}} + Ce^{B\lambda}\left\|\Lambda_{q_1}^{\sharp} - \Lambda_{q_2}^{\sharp}\right\|, \quad \theta \in \mathbb{S}^{d-1} \text{ a.e. } x \in \mathbb{R}^{d}.$$

The proof is then completed. $\square$

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